

# Digital curves and surfaces

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## 1 Digital geometry

In classical Euclidean geometry, the spaces are infinite. A curve is infinitely thin and contains infinitely many points. If we want to represent a geometrical space on a computer, the space must be finite. Digital geometry is the geometry for such spaces.

The most fundamental space for classical geometry is perhaps  $\mathbb{R}^n$  when  $n = 2$  or  $n = 3$ . A natural digital analogue is  $\mathbb{Z}^n$ , where a point is described by  $n$  coordinates and each coordinate is an integer. The computer screen contains small squares of different colors, often called picture elements or pixels for short. A common resolution for the computer screen is  $1024 \times 768$  pixels and then each pixel is addressed by a horizontal coordinate between 0 and 1023, and a vertical coordinate between 0 and 767. In other words: the computer screen is a finite subset of the *digital plane*,  $\mathbb{Z}^2$ . To avoid complications with the boundary, it is often convenient to develop the mathematical theory for all of  $\mathbb{Z}^2$  instead of this finite subset.

What should we mean by a curve in the digital plane? It seems natural to require that a curve is connected in some sense and also that it is a thin set. For this it is necessary to define connectedness in the digital plane. Let us say that two points  $p = (p_1, p_2)$  and  $q = (q_1, q_2)$  in  $\mathbb{Z}^2$  are *8-connected* if  $p \neq q$  and  $\max(|p_1 - q_1|, |p_2 - q_2|) = 1$ . Hence, the points that are 8-connected to  $(0, 0)$  are the eight points

$$(1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0), (-1, -1), (0, -1), \text{ and } (1, -1).$$

An 8-connected curve is then a sequence  $(a_1, a_2, \dots, a_k)$  of points in  $\mathbb{Z}^2$  such that  $a_j$  and  $a_i$  are 8-connected precisely when  $|j - i| = 1$ . An example of such a curve is given by  $a_i = (i, i)$  for  $1 \leq i \leq 100$ . This sequence is a digital line as defined by Rosenfeld (1974).

From classical geometry we know that when two curves cross, then there is a point of intersection. This has to do with the completeness of the real number system. In digital geometry, this property is not automatically true. If  $(a_i)$  is as above and  $(b_i)$  is an 8-connected curve defined by  $b_i = (101 - i, i)$ ,  $1 \leq i \leq 100$ , then  $(a_i)$  and  $(b_i)$  clearly cross, but they have no point in common.

## 2 The Khalimsky topology

We want to define a curve as the image of a continuous mapping  $f: \mathbb{Z} \rightarrow \mathbb{Z}^2$ . To define what it means for a function to be continuous, we need a topology<sup>1</sup> for  $\mathbb{Z}$  and  $\mathbb{Z}^2$ . The Khalimsky

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<sup>1</sup>A topology is a mathematical structure that defines continuity and connectedness in a space. In this section we assume that the reader is familiar with point set topology.

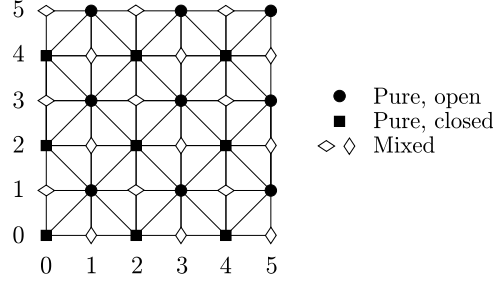


Figure 1: The Khalimsky plane.

topology was studied by Efim Khalimsky in the early 1970's, but became widespread in connection with Khalimsky et al.'s (1990) paper. Here, we shall just mention that an open set on  $\mathbb{Z}$  is given by a union of sets of the type  $[k, l] \cap \mathbb{Z}$  where  $k$  and  $l$  are odd integers, i.e, these sets form a basis for the Khalimsky topology. It is an easy exercise to check that the Khalimsky line is a connected space.

A *Khalimsky interval* is an interval  $[a, b] \cap \mathbb{Z}$  of integers equipped with the topology induced from the Khalimsky line. Let  $X$  be any topological space and  $Y$  be a subset of  $X$ . If  $Y$  is homeomorphic with a Khalimsky interval, then we call  $Y$  a *Khalimsky arc*. The points in  $Y$  which corresponds to the endpoints of the interval are called *endpoints*. The space  $X$  is called *Khalimsky arc-connected* if any two distinct points  $x$  and  $y$  in  $X$  are endpoints of a Khalimsky arc in  $X$ .

In any topological space, the intersection of finitely many open sets is open. If we require than an arbitrary intersection of open sets is open, we obtain a smaller class of topological spaces. For example, all finite topological spaces have this property, while  $\mathbb{R}^n$  with the usual topology does not. Such topological spaces were studied by Alexandrov (1937), and we shall call a space of this type a *smallest-neighborhood space*. The motivation for this name is that intersection of all neighborhoods of a point is again a neighborhood—the smallest neighborhood of the point. Smallest-neighborhood spaces are interesting in the study of digital geometry and the following theorem shows that the Khalimsky topology plays a particular role here.

**Theorem 1.** *A  $T_0$  smallest-neighborhood space is connected if and only if it is Khalimsky arc-connected.*

The Khalimsky plane is  $\mathbb{Z}^2$  with the product topology. In Figure 1, the Khalimsky plane is illustrated. The points that are open in both coordinates are open in the plane and the points that are closed in both coordinates are closed in the plane. A point that is open or closed is called *pure*. Points that are open in one coordinate and closed in the other are called *mixed*.

### 3 Curves and surfaces in Khalimsky spaces

Given a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  we want to find a integer-valued, Khalimsky-continuous approximation  $F: \mathbb{Z}^n \rightarrow \mathbb{Z}$ . Theorem 14 in (Melin 2004b) shows how this can be done if  $f$  satisfies a natural technical condition; it is possible if  $f$  is Lipschitz with Lipschitz constant 1 for the  $l^\infty$ -metric. From a more geometrical point of view, we consider the graph of a real-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ . If  $n = 1$  it is a curve, and in general it is a hyper-surface in  $\mathbb{R}^{n+1}$ . The approximation  $F$  of  $f$  defines a digital curve or surface in  $\mathbb{Z}^{n+1}$ , namely the graph of  $F$ . We call

this digital surface the *Khalimsky-continuous digitization*<sup>2</sup> of the original surface.

In (Melin 2003), the Khalimsky-continuous digitization of straight lines in the plane was studied. It was shown that such a line separates the Khalimsky plane into precisely two connectivity components. A corollary of this is that two non-parallel digital lines has a point in common. This is proved by the following argument: Since the first line separates the plane, it must also separate the other line into (at least) two components, hence there is a point of intersection. We remark, however, that this point is not unique in general; if the lines are almost parallel, the intersection can be of arbitrary high (but finite) cardinality. In three dimensions, we have the following analogue, proved in (Melin 2004a).

**Theorem 2.** *Let  $U$  and  $V$  be the Khalimsky-continuous digitizations of two non-parallel planes in  $\mathbb{R}^3$ . Then the intersection contain a subset homeomorphic to  $\mathbb{Z}$  with the Khalimsky topology.*

In other words, the intersection of two planes contain a line. This is what we expect from the continuous case, but a corresponding theorem does not exist for other types of digital planes.

## References

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<sup>2</sup>Digitize means to convert, for example, a picture into a digital form that can be processed by a computer.